<u>**MA209   Eksamen Sommer 2008**</u>

Poeng:

Oppg nr			Delsum
1	a	3	
	b	3	
	c	3	
	d	3	
	e	3	
	f	6	21
2	a	3	
	b	3	
	c	3	
	d	3	12
3		6	6
4	a	3	
	b	3	
	c	3	9
Sum			48

LYKKE TIL !

1. Vi har gitt følgende kurve C i rommet:

$$\vec{r}(t) = \cos(t)\vec{i} + \sin(t)\vec{j} + \sqrt{2}\sin(t)\vec{k} = [\cos(t), \sin(t), \sqrt{2}\sin(t)] \qquad 0 \le t \le 2\pi$$

a) Bestem hastighetsvektoren og akselerasjonsvektoren.

b) Finn enhetstangent-vektoren til kurven i punktet:

$$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1\right).$$

c) Bestem kurvens krumning.

d) Kurven ligger i et plan parallell med x-aksen.
Finn ligningen for dette planet.

Vi har gitt vektor-feltet:

$$\vec{F}(x, y, z) = xyz\vec{k} = [0, 0, xyz]$$

e) Beregn divergens ($div\vec{F}$) og curl ($curl\vec{F}$) til det gitte vektor-feltet.

f) Avgjør om vektor-feltet er et konservativt felt (gradientfelt).

Beregn

$$\int_C \vec{F} \cdot d\vec{r}$$

både direkte og ved hjelp av Stokes teorem.

2. Vi har gitt flatene S_1 (del av en kuleflate) og S_2 (del av en sideflate i kjegle) ved:

S_1 $\quad x^2 + y^2 + (z-3)^2 = 4 \qquad 1 \le z \le 3 - \sqrt{2}$

S_2 $\quad z = 3 - \sqrt{x^2 + y^2}$

a) Vis at skjæringskurven mellom S_1 og S_2 er gitt ved:

$x^2 + y^2 = 2 \qquad z = 3 - \sqrt{2}$

b) Bestem vha trippel-integral volumet av romlegemet avgrenset av S_1 og S_2

Vi har gitt følgende vektorfelt:

$$\vec{F}(x,y,z) = (x+yz)\vec{i} + (y-xz)\vec{j} + (z+yx)\vec{k} = [x+yz, y-xz, z+yx]$$

c) Bestem $\nabla \cdot \vec{F}$

d) La S være overflaten til romlegemet beskrevet i b).
Finn

$$\iint_S \vec{F} \cdot \vec{n} dS$$

der $\vec{n}$ er ytre enhetsnormal til S.

3. Bestem z = z(x,y) fra følgende partielle differensialligning:

$$\frac{\partial^2 z}{\partial x \partial y} = 8x^3 y$$
$$z(x,0) = 2x$$
$$z(0.y) = 3y$$

4. En 50 cm lang, tynn jernstav med diffusivitet k = 0.15 cm^2/s initieres med en temperatur lik 0 ^{0}C i venstre ende og som øker lineært opp til temperaturen 100 ^{0}C ved stavens høyre ende.

Ved tiden t = 0 varme-isoleres jerstavens sideflate, samtidig som stavens endepunkter omgis av is slik at begge endepunktene holdes konstant på temperatur 0 ^{0}C.

a) Bestem funksjonen f(x) som gir temperatur-initieringen.

b) Bestem Fourier sinus-rekken til f(x).

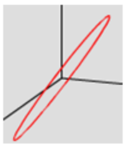
c) Bruk resultatet fra oppg b) til å bestemme temperaturen etter 3 minutter 40 cm fra stavens venstre ende.
Beregn resultatet ved å ta med 3 ledd i rekke-uttrykket for temperaturen.

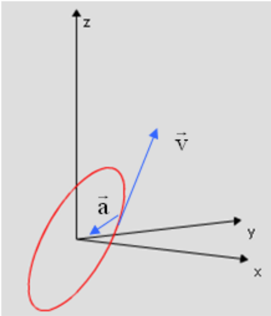
Løsning:

1. a) Hastighet- og akselerasjons-vektor:

$$\vec{v}(t) = \vec{r}\,'(t) = \left[-\sin(t), \cos(t), \sqrt{2}\cos(t)\right]$$

$$\vec{a}(t) = \vec{r}\,''(t) = \left[-\cos(t), -\sin(t), -\sqrt{2}\sin(t)\right]$$





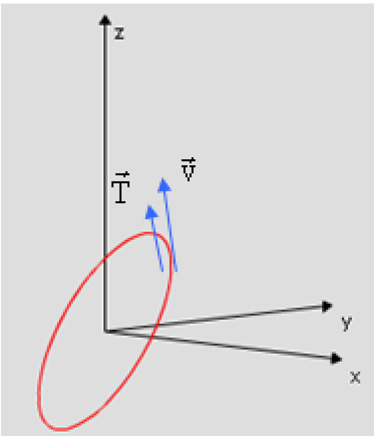
b) Enhetstangent-vektor:

$$\vec{v}(t) = \vec{r}\,'(t) = \left[-\sin(t), \cos(t), \sqrt{2}\cos(t)\right]$$

$$|\vec{v}(t)| = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + (\sqrt{2}\cos(t))^2} = \sqrt{1 + 2\cos^2(t)}$$

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{\left[-\sin(t), \cos(t), \sqrt{2}\cos(t)\right]}{\sqrt{1 + 2\cos^2(t)}}$$

$$\vec{T}\left(\frac{\pi}{4}\right) = \frac{\left[-\sin\left(\frac{\pi}{4}\right), \cos\left(\frac{\pi}{4}\right), \sqrt{2}\cos\left(\frac{\pi}{4}\right)\right]}{\sqrt{1 + 2\cos^2\left(\frac{\pi}{4}\right)}} = \frac{\left[\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}, 1\right]}{\sqrt{2}} = \left[-\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right]$$



c) Kurvens krumning:

$$\kappa(t) = \frac{|\vec{v}(t) \times \vec{a}(t)|}{|\vec{v}(t)^3|} = \frac{\sqrt{3}}{(1 + 2\cos^2(t))^{\frac{3}{2}}}$$

d)

$$y = \sin(t)$$
$$z = \sqrt{2}\sin(t)$$
$$\Downarrow$$
$$z = \sqrt{2}y$$

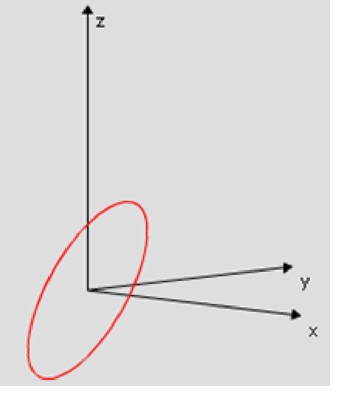
Herav : Kurven C ligger i planet $z = \sqrt{2}y$

e) Divergens

$$\text{div}\vec{F} = \nabla \cdot \vec{F} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot [0, 0, xyz] = 0 + 0 + xy = \underline{\underline{xy}}$$

Curl:

$$\text{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & xyz \end{vmatrix} = \underline{\underline{[xz, -yz, 0]}}$$



f)

$$\text{curl}\vec{F} = [xz, -yz, 0] \neq \vec{0} \quad \Rightarrow \quad \text{Vektorfeltet er ikke et konservativt felt}$$

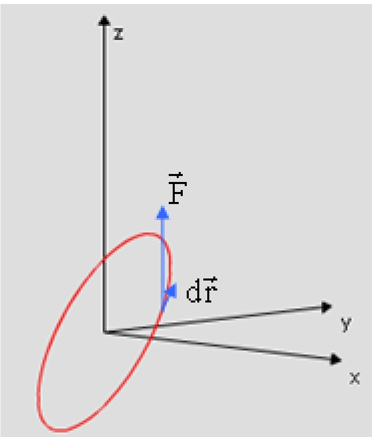
Direkte (uten Stokes teorem):

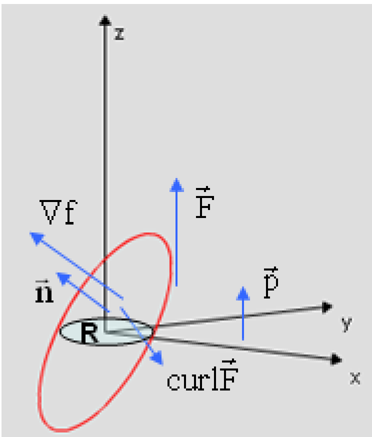
$$\int_C \vec{F}\cdot d\vec{r} = \int_C [0,0,xyz]\cdot\left[-\sin(t),\cos(t),\sqrt{2}\cos(t)\right]dt$$
$$= \int_0^{2\pi}\left[0,0,\cos(t)\sin(t)\sqrt{2}\sin(t)\right]\cdot\left[-\sin(t),\cos(t),\sqrt{2}\cos(t)\right]dt$$
$$= \int_0^{2\pi} 2\sin^2(t)\cos^2(t)dt$$
$$= \int_0^{2\pi}\frac{1}{2}\sin^2(2t)dt$$
$$= \int_0^{2\pi}\frac{1}{4}(1-\cos(4t)dt = \left[\frac{1}{4}(t-\frac{1}{4}\sin(4t)\right]_0^{2\pi} = \underline{\underline{\frac{\pi}{2}}}$$

Ved hjelp av Stokes teorem:

$$f(x,y,z) = z - \sqrt{2}y = 0$$
$$\nabla f = \left[0,-\sqrt{2},1\right]$$

$$\iint_S (\text{curl}\vec{F}\cdot\vec{n})dS = \iint_R [xz,-yz,0]\cdot\frac{\nabla f}{|\nabla f|}\frac{|\nabla f|}{|\nabla f\cdot\vec{p}|}dxdy = \iint_R [xz,-yz,0]\cdot\frac{\nabla f}{|\nabla f|}\frac{|\nabla f|}{|\nabla f\cdot[0,0,1]|}dxdy$$
$$= \iint_R [xz,-yz,0]\cdot\frac{\left[0,-\sqrt{2},1\right]}{1}dxdy$$
$$= \iint_R 2y^2dxdy$$
$$= \int_0^{2\pi}\int_0^1 2(r\sin(t))^2 rdrdt$$
$$= \int_0^{2\pi}\int_0^1 2r^3\sin^2(t)rdrdt$$
$$= \int_0^{2\pi}\frac{1}{2}(1-\cos(2t)dt\cdot\int_0^1 2r^3dr$$
$$= \left[\frac{1}{2}(t-\frac{1}{2}\sin(2t)\right]_0^{2\pi}\cdot\left[\frac{1}{4}r^4\right]_0^1 = \pi\cdot\frac{1}{2} = \underline{\underline{\frac{\pi}{2}}}$$





2. a) Skjæringskurven mellom S_1 og S_2

$$z = 3 - \sqrt{x^2 + y^2} \qquad\qquad x^2 + y^2 + (z-3)^2 = 4$$

$$\Downarrow$$

$$(z-3)^2 = x^2 + y^2$$

$$x^2 + y^2 + x^2 + y^2 = 4$$

$$2(x^2 + y^2) = 4$$

$$x^2 + y^2 = 2 \qquad z = 3 - \sqrt{x^2 + y^2} = 3 - \sqrt{2}$$

$$x^2 + y^2 = 2 \qquad z = 3 - \sqrt{2}$$

b)

$$V = \iiint_V dV = \iiint_V dxdydz$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}}\int_{3-\sqrt{4-r^2}}^{3-r} dzrdrd\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} [z]_{3-\sqrt{4-r^2}}^{3-r} rdrd\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} \left[(3-r) - (3-\sqrt{4-r^2})\right] rdrd\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} \left[\sqrt{4-r^2} - r\right] rdrd\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} \left[r\sqrt{4-r^2} - r^2\right] drd\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{3}(4-r^2)^{\frac{3}{2}} - \frac{1}{3}r^3\right]_0^{\sqrt{2}} d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{3}\cdot 2^{\frac{3}{2}} + \frac{1}{3}\cdot 8 - \frac{1}{3}\cdot 2^{\frac{3}{2}}\right] d\theta$$

$$= \int_0^{2\pi} \left[-\frac{4}{3}\sqrt{2} + \frac{8}{3}\right] d\theta$$

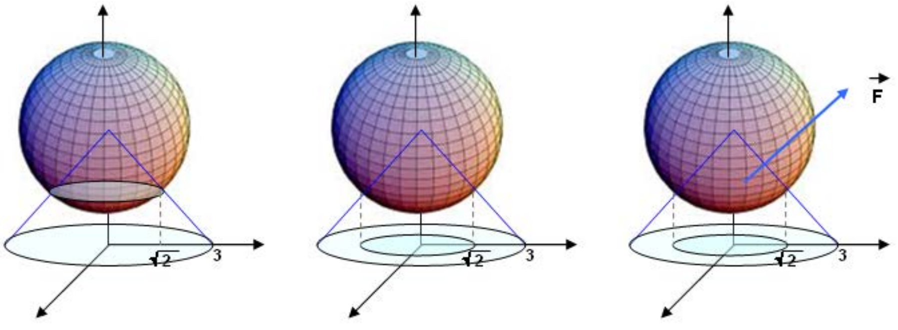
$$= 2\pi \cdot \left[-\frac{4}{3}\sqrt{2} + \frac{8}{3}\right] = \underline{\underline{\frac{8}{3}\pi(2-\sqrt{2})}}$$

c)

$$\nabla \cdot \vec{F} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot [x + yz, y - xz, z + yx] = 1 + 1 + 1 = \underline{\underline{3}}$$

d)

$$\iint_S \vec{F} \cdot \vec{n} dS = \iiint_V \nabla \cdot \vec{F} dV = \iiint_V 3dV = 3\iiint_V dV = 3V = 3 \cdot \frac{8}{3}\pi(2-\sqrt{2}) = \underline{\underline{8\pi(2-\sqrt{2})}}$$

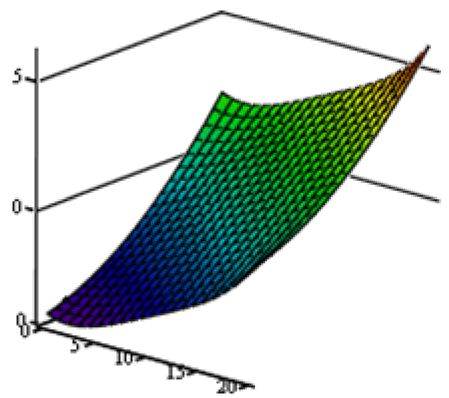


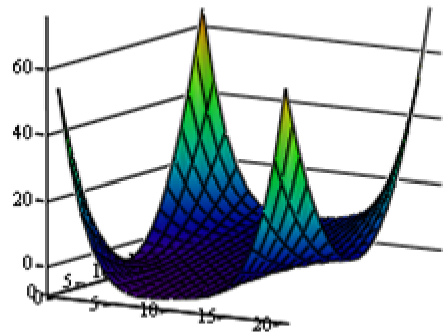
3.

$$\frac{\partial^2 z}{\partial x \partial y} = 8x^3 y$$
$$z(x,0) = 2x$$
$$z(0.y) = 3y$$

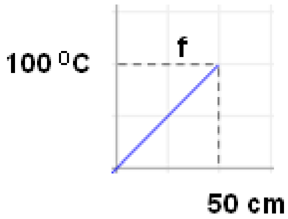
$$\frac{\partial^2 z}{\partial x \partial y} = 8x^3 y$$
$$\int \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) dx = \int 8x^3 y dx$$
$$\frac{\partial z}{\partial y} = 2x^4 y + F(y)$$
$$z(x,y) = x^4 y^2 + \int F(y) dy + G(x) = \underline{x^4 y^2 + H(y) + G(x)}$$

$$z(x,0) = 2x \Rightarrow G(x) = 2x - H(0)$$
$$z(x,y) = x^4 y^2 + H(y) + 2x - H(0)$$
$$z(0,y) = 3y \Rightarrow H(y) = 3y + H(0)$$
$$\underline{\underline{z(x,y) = x^4 y^2 + 3y + 2x}}$$





4. a) f(x)=Ax hvor A = 2



b) Fourier sinus-rekke

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

hvor

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

Med f(x) = Ax får vi videre:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

hvor

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx = \frac{2}{L}\int_0^L Ax\sin\frac{n\pi x}{L}dx$$

$$b_n = \frac{2}{L}\int_0^L Ax\sin\frac{n\pi x}{L}dx = \frac{2A}{L}\int_0^L x\sin\frac{n\pi x}{L}dx$$

$$= \frac{2AL}{n^2\pi^2}\int_0^{n\pi} u\sin(u)du \qquad \text{substitusjon} \quad u = \frac{n\pi x}{L}$$

$$= \frac{2AL}{n^2\pi^2}\left[\left[-u\cos(u)\right]_0^{no} - \int_0^{n\pi}\cos(u)du\right] \qquad \text{delvis integrasjon}$$

$$= \frac{2AL}{n^2\pi^2}\left[-u\cos(u) + \sin(u)\right]_0^{n\pi}$$

$$= -\frac{2AL}{n^2\pi^2}\left[n\pi\cos(n\pi)\right]$$

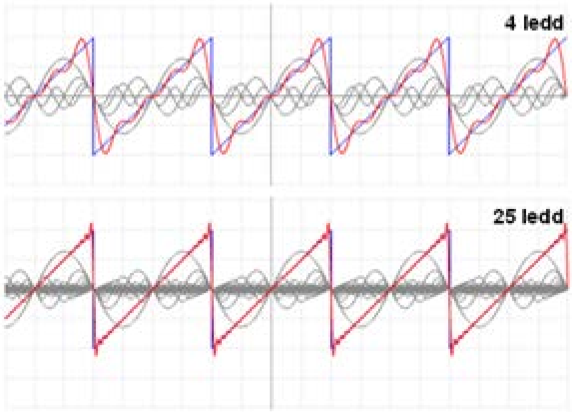
$$= -\frac{2AL}{n\pi}\left[\cos(n\pi)\right]$$

$$= -\frac{2AL}{n\pi}(-1)^n$$

$$= \frac{2AL}{n\pi}(-1)^{n+1}$$

$$b_n = \frac{2AL}{n\pi}(-1)^{n+1}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L} = \sum_{n=1}^{\infty}\left[\frac{2AL}{n\pi}(-1)^{n+1}\right]\sin\frac{n\pi x}{L}$$



c) Fra varmeligningen har vi nå:

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2 kt}{L^2}} \sin\frac{n\pi x}{L}$$

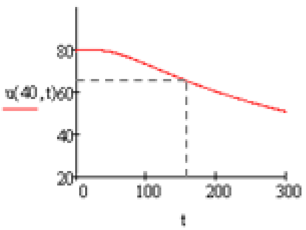
hvor

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

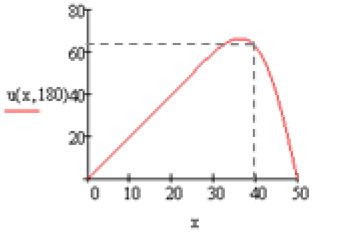
$$= \frac{2AL}{n\pi}(-1)^{n+1}$$

3 ledd i rekken gir : u(40,180) = 61.134
50 ledd i rekken gir : u(40,180) = 62.643

$$u(x,t) := \sum_{n=1}^{200}\left[\frac{(2\cdot 2\cdot 50)}{(n\cdot\pi)}\cdot(-1)^{n+1}\cdot e^{-n^2\cdot\pi^2\cdot 0.15\cdot\frac{t}{50^2}}\cdot\sin\left(n\cdot\pi\cdot\frac{x}{50}\right)\right]$$



Temperaturen u
som funksjon av tiden t
for x = 40 cm



Temperaturen u
som funksjon av posisjon x
for t = 180 sekunder