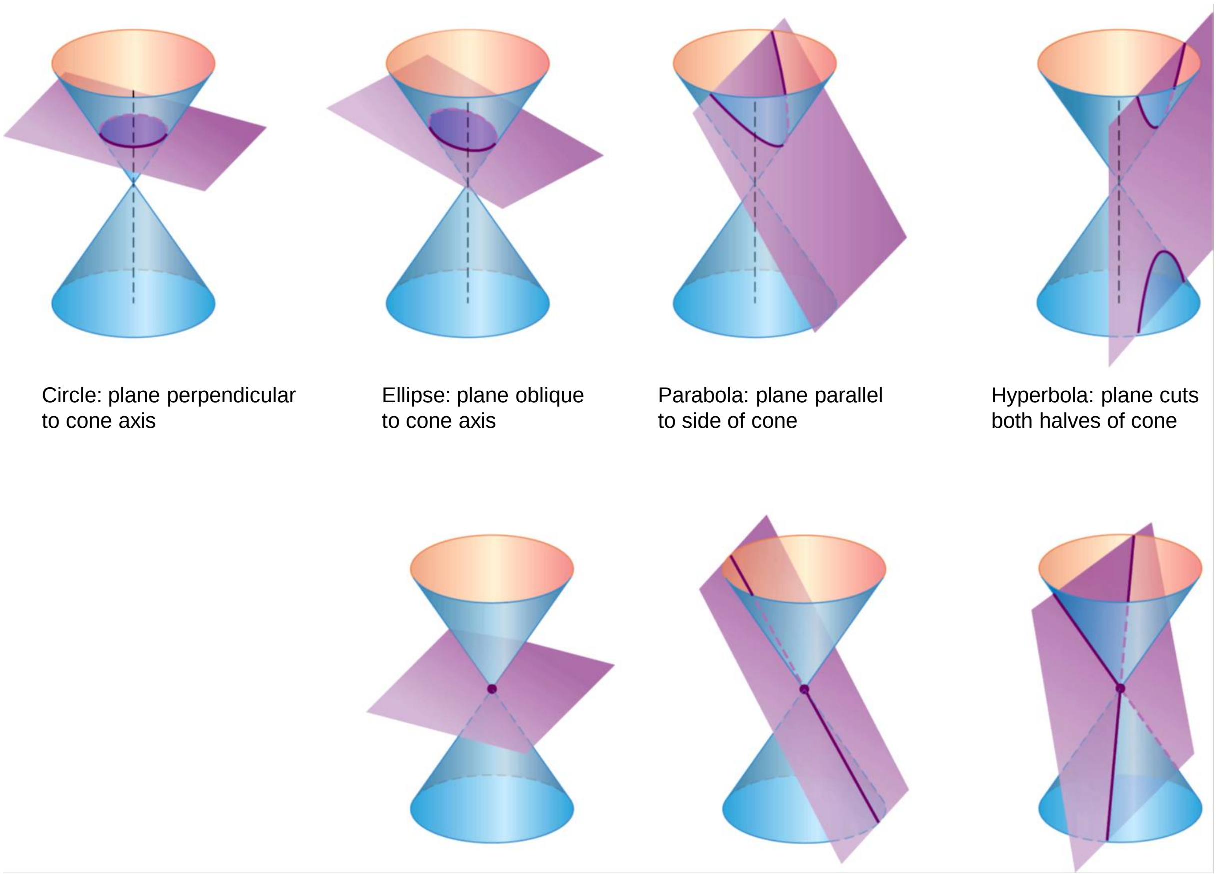
MA-209

Formelhefte

Per Henrik Hogstad

Universitetet i Agder

Kjeglesnitt



Circle: plane perpendicular to cone axis

Ellipse: plane oblique to cone axis

Parabola: plane parallel to side of cone

Hyperbola: plane cuts both halves of cone

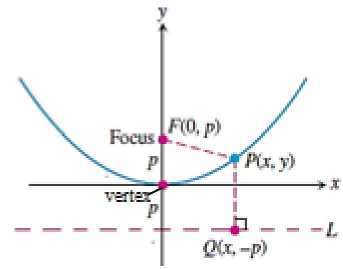
Point: plane through cone vertex only

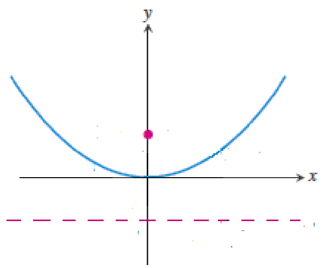
Single line: plane tangent to cone

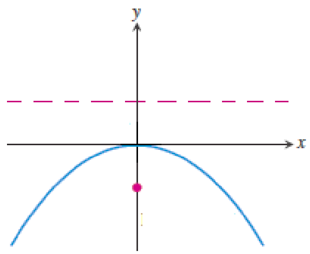
Pair of intersecting lines

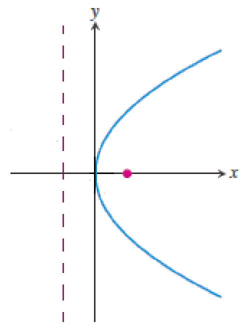
Kvadratisk kurve	$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$	$B^2 - 4AC = 0$	**Parabel**
		$B^2 - 4AC < 0$	**Ellipse**
		$B^2 - 4AC > 0$	**Hyperbel**

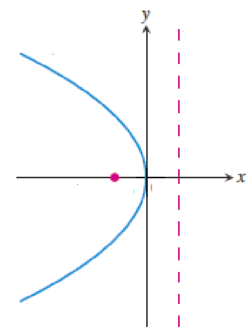
Kjeglesnitt - Parabel

Parabel	$y = \frac{1}{4p}x^2$ $PF = PQ$	





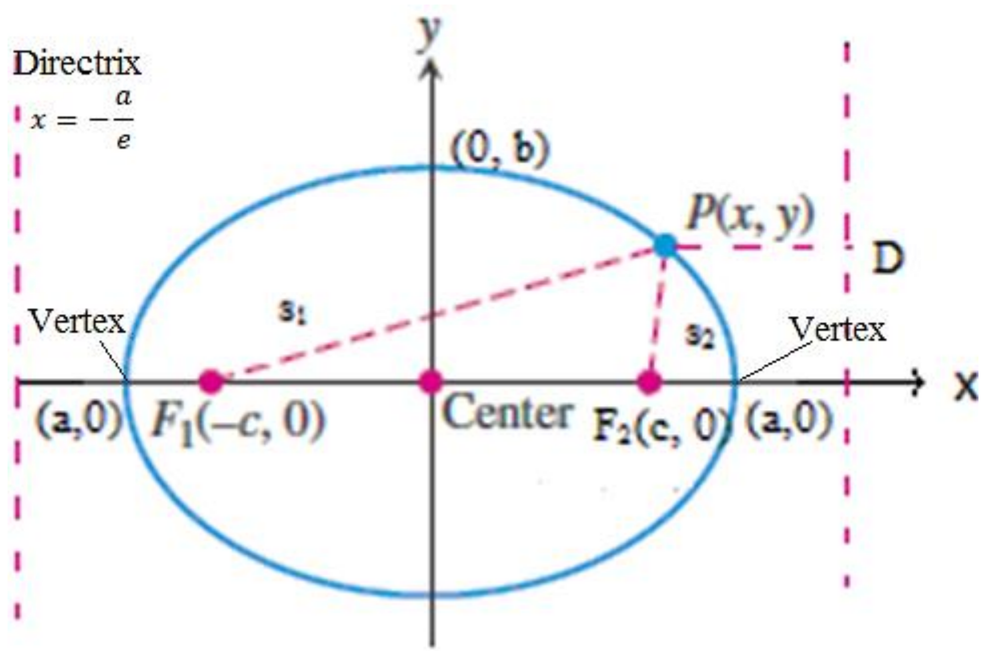




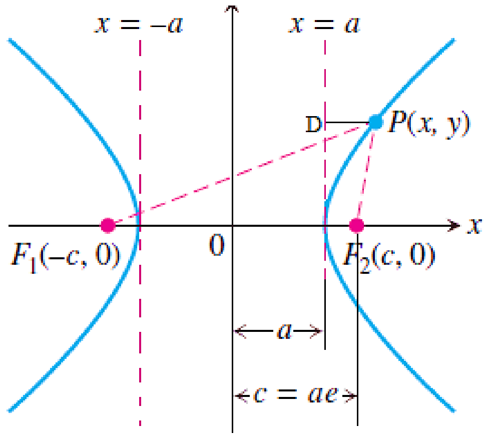
	$x^2 = 4py$	$x^2 = -4py$	$y^2 = 4px$	$y^2 = -4px$
Fokus	$(0, p)$	$(0, -p)$	$(p, 0)$	$(-p, 0)$
Styrelinje	$y = -p$	$y = p$	$x = -p$	$x = p$

Eksentrisitet	$e = \frac{PF}{PQ} = 1$	

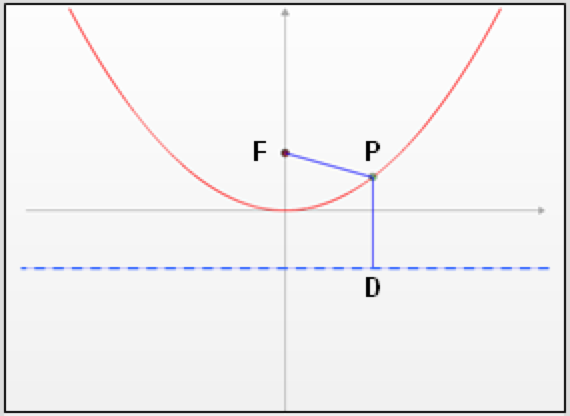
Kjeglesnitt - Ellipse

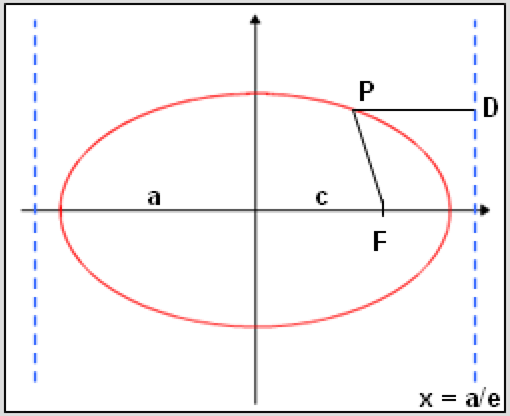
Ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $s_1+s_2=2a$	
		
Eksentrisitet	$e=\frac{PF}{PQ}=\frac{c}{a}<1$	

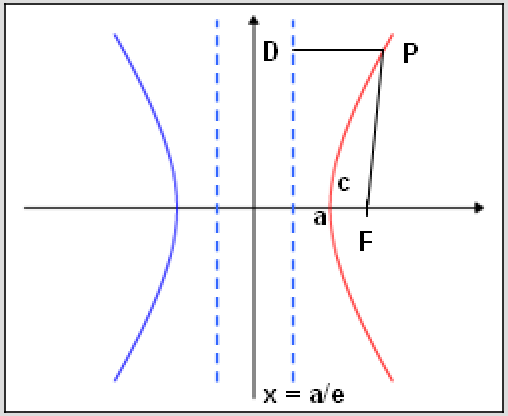
Kjeglesnitt - Hyperbel

Hyperbel	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $PF_1 - PF_2 = \pm 2a$	
Eksentrisitet	$e = \frac{PF}{PD} = \frac{c}{a} > 1$	

Eksentrisitet





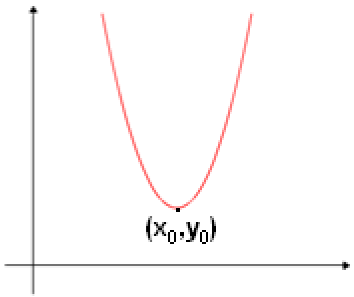


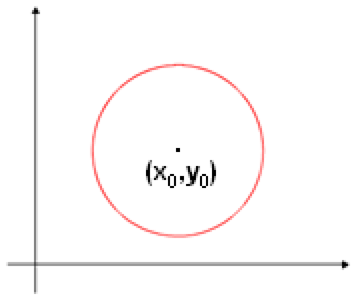
$$e = \frac{PF}{PD} = 1$$

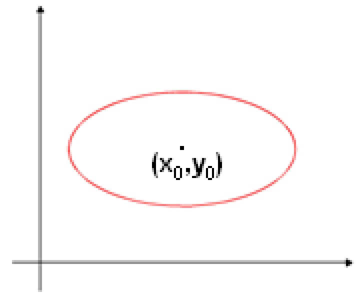
$$e = \frac{PF}{PD} = \frac{c}{a} < 1$$

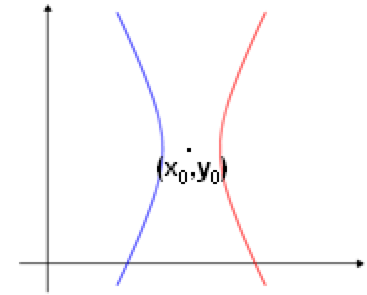
$$e = \frac{PF}{PD} = \frac{c}{a} > 1$$

$$e = \frac{PF}{PD} = \frac{\text{Avstand fra P til nærmeste fokuspunkt}}{\text{Avstand fra P til nærmeste styrelinje}}$$





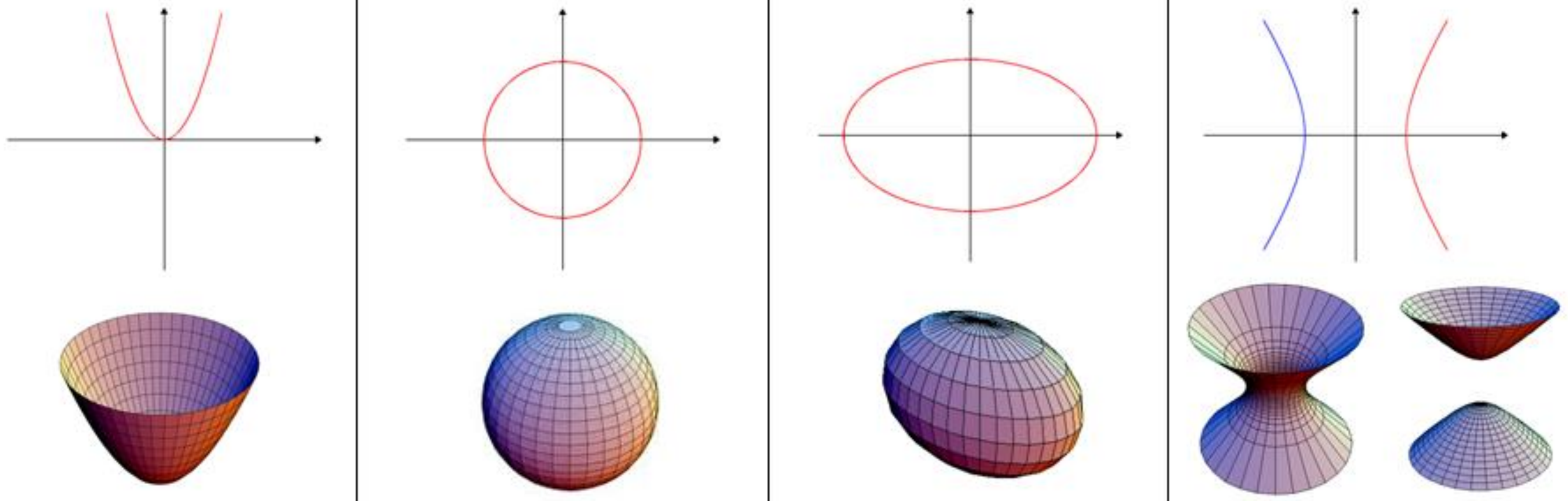




Parabel	Sirkel	Ellipse	Hyperbel
$y = y_0 + k(x - x_0)^2$	$(x - x_0)^2 + (y - y_0)^2 = R^2$	$\left(\frac{x - x_0}{a}\right)^2 + \left(\frac{y - y_0}{b}\right)^2 = 1$	$\left(\frac{x - x_0}{a}\right)^2 - \left(\frac{y - y_0}{b}\right)^2 = 1$
$B^2 - 4AC = 0$	$A = C \qquad B = 0$	$B^2 - 4AC < 0$	$B^2 - 4AC > 0$

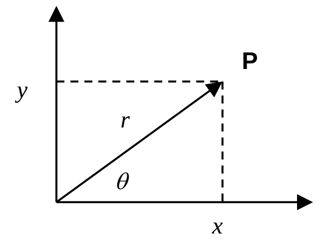
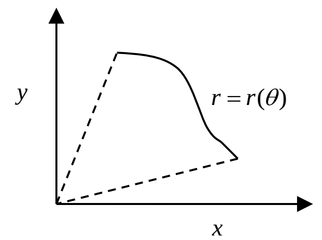
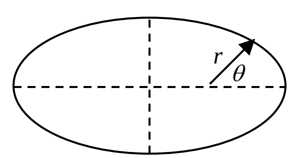
$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

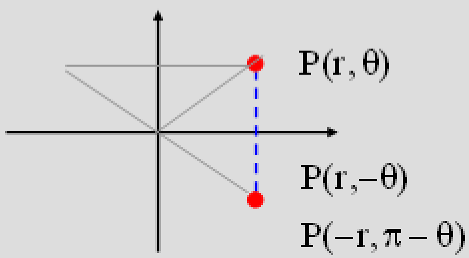
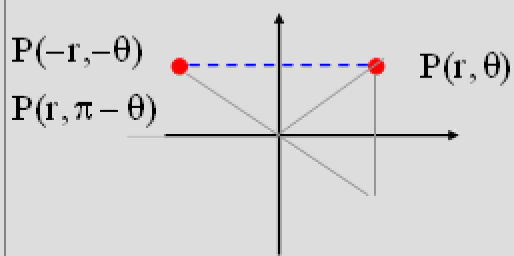
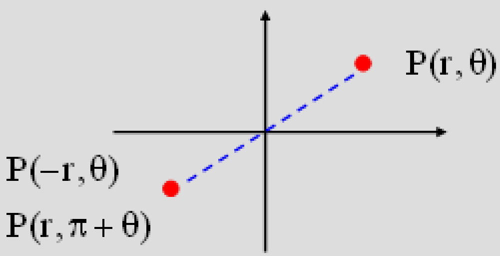
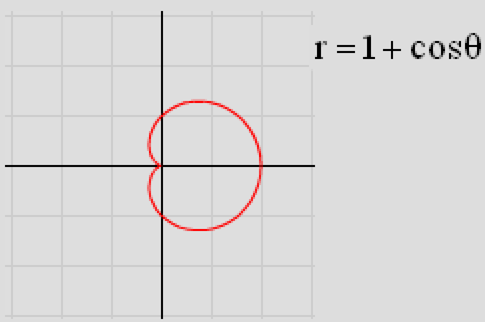
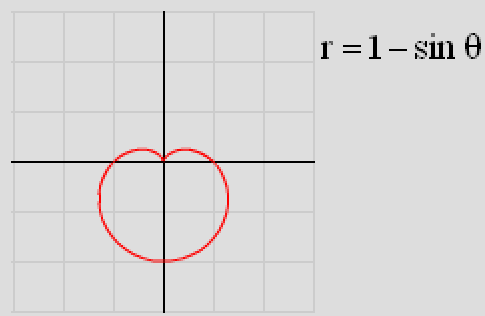
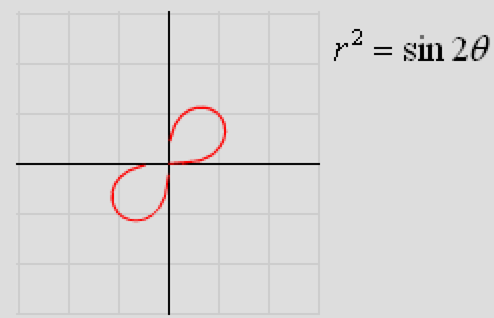
Paraboloide – Kule – Ellipsoide – Hyperboloide



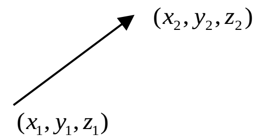
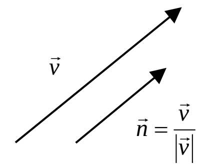
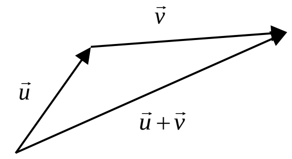
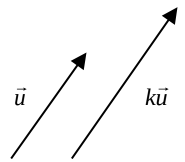
Paraboloide	Kule	Ellipsoide	Hyperboloide
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-z=0$	$x^2+y^2+z^2=R^2$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=\pm 1$

Polar-koordinater

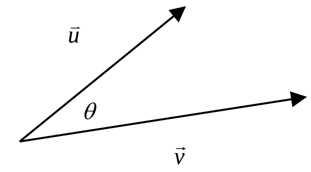
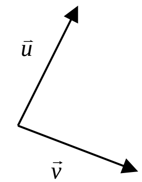
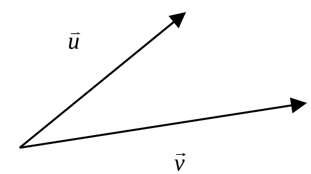
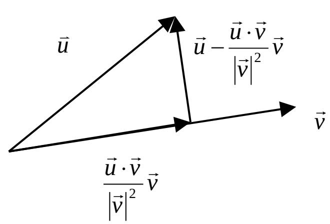
Koordinater	$x = r\cos\theta$ $y = r\sin\theta$ $x^2 + y^2 = r^2$	
Areal	$A = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 d\theta$	
Lengden av polar kurve	$L = \int_{\theta_2}^{\theta_1} \sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta$	
Polar ligning for konisk seksjon	$r = \frac{ke}{1 + e\cos\theta}$	$e < 1$ Ellipse $e = 1$ Parabel $e > 1$ Hyperbel
Ellipse	$r = \frac{a(1-e^2)}{1 + e\cos\theta}$	

Symmetri om x-aksen:	Symmetri om y-aksen:	Symmetri om origo:
$r(-\theta) = r(\theta) \quad \vee \quad r(\pi - \theta) = -r(\theta)$	$r(-\theta) = -r(\theta) \quad \vee \quad r(\pi - \theta) = r(\theta)$	$r(\theta) = -r(\theta) \quad \vee \quad r(\pi + \theta) = r(\theta)$
		
		

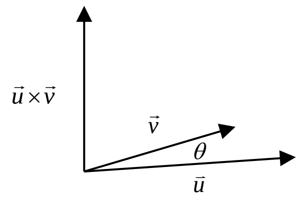
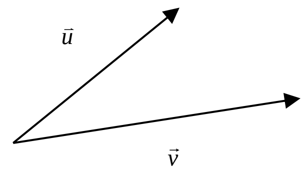
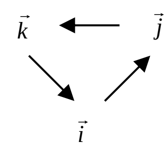
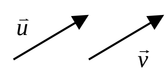
Vektorer og geometri i rommet - Vektor

Vektor	$\vec{v} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$ $= [x_2 - x_1, y_2 - y_1, z_2 - z_1]$ $= [v_1, v_2, v_3]$	
Lengden av en vektor	$\lvert\vec{v}\rvert = \sqrt{v_1^{\,2} + v_2^{\,2} + v_3^{\,2}}$	
Enhetsvektor	$\vec{n} = \frac{\vec{v}}{\lvert\vec{v}\rvert}$	
Addisjon	$\vec{u} = [u_1, u_2, u_3]$ $\vec{v} = [v_1, v_2, v_3]$ $\vec{u} + \vec{v} = [u_1 + v_1, u_2 + v_2, u_3 + v_3]$	
Multiplikasjon med skalar	$\vec{u} = [u_1, u_2, u_3]$ $k\vec{u} = [ku_1, ku_2, ku_3]$	
Regneregler for vektorer	$\vec{u} + \vec{v} = \vec{v} + \vec{u}$ $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ $\vec{u} + \vec{0} = \vec{u}$ $\vec{u} + (-\vec{u}) = \vec{0}$ $0\vec{u} = \vec{0}$ $1\vec{u} = \vec{u}$ $a(b\vec{u}) = (ab)\vec{u}$ $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$ $(a + b)\vec{u} = a\vec{u} + b\vec{u}$	

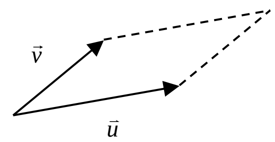
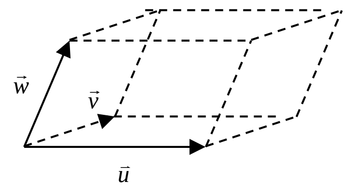
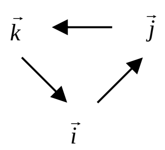
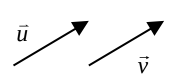
Vektorer og geometri i rommet - Skalarprodukt

Skalarprodukt	$\vec{u}\cdot\vec{v}=\lvert\vec{u}\rvert\lvert\vec{v}\rvert\cos\theta=u_1v_1+u_2v_2+u_3v_3$	
Regler	$\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$ $(c\vec{u})\cdot\vec{v}=\vec{u}\cdot(c\vec{v})$ $\vec{u}\cdot(\vec{v}+\vec{w})=\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$ $\vec{u}\cdot\vec{u}=\lvert\vec{u}\rvert^2$ $\vec{0}\cdot\vec{u}=0$	
Ortogonale vektorer	$\vec{u}\perp\vec{v}\quad\Leftrightarrow\quad\vec{u}\cdot\vec{v}=0$	
Projeksjon av u-vektor på v-vektor **Skalarkomponent av u-vektor i retning v-vektor**	$proj_{\vec{v}}\vec{u}=\frac{\vec{u}\cdot\vec{v}}{\lvert\vec{v}\rvert^2}\vec{v}$ $\lvert\vec{u}\rvert\cos\theta=\vec{u}\cdot\frac{\vec{v}}{\lvert\vec{v}\rvert}$	
Vektor skrevet som en sum av ortogonale vektorer	$\vec{u}=\underbrace{\frac{\vec{u}\cdot\vec{v}}{\lvert\vec{v}\rvert^2}\vec{v}}_{\text{Parallell med }\vec{v}}+\underbrace{\left(\vec{u}-\frac{\vec{u}\cdot\vec{v}}{\lvert\vec{v}\rvert^2}\vec{v}\right)}_{\text{Normal på }\vec{v}}$	

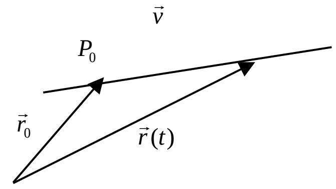
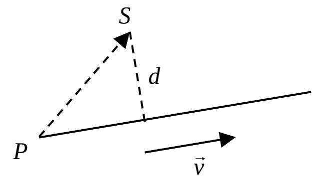
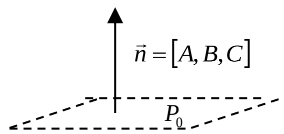
Vektorer og geometri i rommet - Vektorprodukt

Vektorprodukt	$\vec{u}\times\vec{v}=\lvert\vec{u}\rvert\lvert\vec{v}\rvert\sin\theta\,\vec{n}$ $=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}$ $=\left[u_2v_3-u_3v_2,u_3v_1-u_1v_3,u_1v_2-u_2v_1\right]$	
Regler	$(r\vec{u})\times(s\vec{v})=(rs)\vec{u}\times\vec{v}$ $\vec{u}\times(\vec{v}+\vec{w})=\vec{u}\times\vec{v}+\vec{u}\times\vec{w}$ $(\vec{v}+\vec{w})\times\vec{u}=\vec{v}\times\vec{u}+\vec{w}\times\vec{u}$ $\vec{v}\times\vec{u}=-(\vec{u}\times\vec{v})$ $\vec{0}\times\vec{u}=\vec{0}$	
	$\vec{i}\times\vec{j}=\vec{k}$ $\vec{j}\times\vec{k}=\vec{i}$ $\vec{k}\times\vec{i}=\vec{j}$	
Parallelle vektorer	$\vec{u}\parallel\vec{v}\quad\Leftrightarrow\quad\vec{u}\times\vec{v}=\vec{0}$	

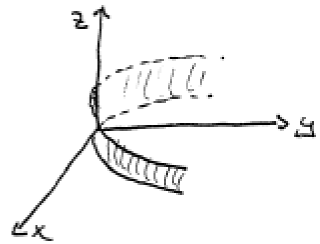
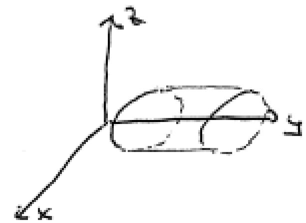
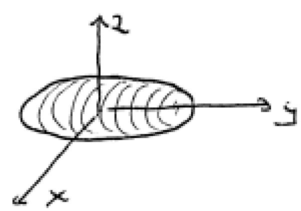
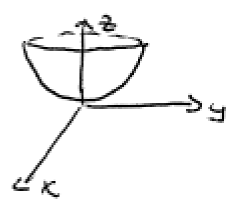
Vektorer og geometri i rommet - Areal - Volum

Areal	$A = \lvert \vec{u} \times \vec{v} \rvert$	
Volum	$V = \lvert (\vec{u} \times \vec{v}) \cdot \vec{w} \rvert$ $= \left\lvert \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \right\rvert$	
	$\vec{i} \times \vec{j} = \vec{k}$ $\vec{j} \times \vec{k} = \vec{i}$ $\vec{k} \times \vec{i} = \vec{j}$	
Parallelle vektorer	$\vec{u} \parallel \vec{v} \iff \vec{u} \times \vec{v} = \vec{0}$	

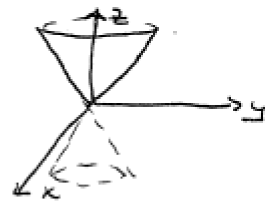
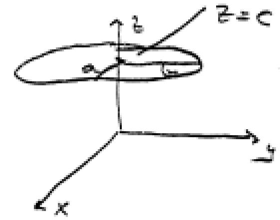
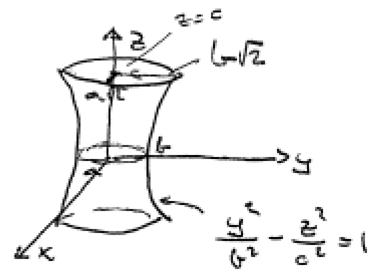
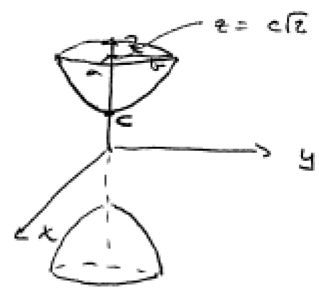
Vektorer og geometri i rommet - Linjer - Plan

Linje **gjennom P_0** **parallell med v-vektor**	$\vec{r}(t) = \vec{r}_0 + t\vec{v}$ $[x, y, z] = [x_0, y_0, z_0] + t[v_1, v_2, v_3]$ $x = x_0 + tv_1$ $y = y_0 + tv_2$ $x = z_0 + tv_3$	
Avstand **fra et punkt S** **til en linje gjennom P** **parallell med v-vektor**	$d = \frac{\lvert \overrightarrow{PS} \times \vec{v} \rvert}{\lvert \vec{v} \rvert}$	
Plan **gjennom P0(x0,y0,z0)** **med normalvektor** $\vec{n} = [A, B, C]$	$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$ $Ax + By + Cz = D$ $\quad D = Ax_0 + By_0 + Cz_0$	

Vektorer og geometri i rommet - Kvadratiske flater

Kvadratiske flater	$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Jz + K = 0$	
Parabolsk sylinder	$y = x^2$	
Elliptisk sylinder	$\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1$	
Ellipsoide	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	
Elliptisk paraboloide	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$	

Vektorer og geometri i rommet - Kvadratiske flater

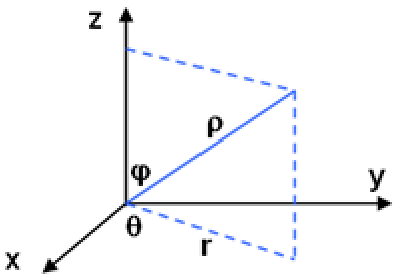
Elliptisk kjegle	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$	 
Hyperboloide	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ **En flate**	
	$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ **To flater**	

Vektor-funksjoner

Posisjon **Hastighet** **Akselerasjon**	$\vec{r}(t)$ $\vec{v}(t)=\vec{r}\,'(t)=\frac{d\vec{r}(t)}{dt}$ $\vec{a}(t)=\vec{v}\,'(t)=\vec{r}\,''(t)=\frac{d^2\vec{r}(t)}{dt^2}$	**Fart** $\frac{ds}{dt}=v(t)=\lvert\vec{v}(t)\rvert$
Enhets tangent vektor	$\vec{T}=\frac{d\vec{r}}{ds}=\frac{\vec{v}}{\lvert\vec{v}\rvert}$	
Krumning **Hoved enhets normal** **Binormal** **Torsjon** **Akselerasjons-komponenter**	$\kappa=\left\lvert\frac{\mathrm{d}\vec{\mathrm{T}}}{\mathrm{ds}}\right\rvert=\frac{1}{\lvert\vec{\mathrm{v}}\rvert}\left\lvert\frac{\mathrm{d}\vec{\mathrm{T}}}{\mathrm{dt}}\right\rvert=\frac{\lvert\vec{\mathrm{v}}\times\vec{\mathrm{a}}\rvert}{\lvert\vec{\mathrm{v}}\rvert^3}$ $\vec{N}=\frac{1}{\kappa}\frac{dT}{ds}=\frac{\frac{d\vec{T}}{dt}}{\left\lvert\frac{d\vec{T}}{dt}\right\rvert}$ $\vec{B}=\vec{T}\times\vec{N}$ $\tau=-\frac{d\vec{B}}{ds}\cdot N=\frac{\begin{vmatrix}\dot{x} & \dot{y} & \dot{z}\\ \ddot{x} & \ddot{y} & \ddot{z}\\ \dddot{x} & \dddot{y} & \dddot{z}\end{vmatrix}}{\lvert\vec{v}\times\vec{a}\rvert^2}$ $\vec{a}=a_T\vec{T}+a_N\vec{N}$	$a_T=\frac{d}{dt}\lvert\vec{v}\rvert$ $a_N=\kappa\lvert\vec{v}\rvert^2=\sqrt{\lvert\vec{a}\rvert^2-a_T{}^2}$

Koordinatsystemer

Sylindrisk → Rektangulær	$x = r\cos\theta$ $y = r\sin\theta$ $z = z$
Sfærisk → Rektangulær	$x = \rho\sin\varphi\cos\theta$ $y = \rho\sin\varphi\sin\theta$ $z = \rho\cos\varphi$
Sfærisk → Sylindrisk	$r = \rho\sin\varphi$ $\theta = \theta$ $z = \rho\cos\varphi$
Rektangulær **Sylindrisk** **Sfærisk**	$dV = dxdydz$ $dV = dzrdrd\theta$ $dV = \rho^2\sin\varphi d\rho d\varphi d\theta$



Vektor-felt - Del-operator

Funksjon	$f = f(x,y,z)$
Vektor-felt	$\vec{F} = [F_1, F_2, F_3]$
Potensial-funksjon f	$\vec{F} = \nabla f$
Del-operator	$\nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]$
Gradient	$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$
Divergens	$\text{div}\vec{F} = \nabla \cdot F = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot [F_1, F_2, F_3] = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$
Curl	$\text{curl}\vec{F} = \nabla \times F = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \times [M, N, P] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]$

Kurve-integral

Kurve-integral	$\int_C f(x,y,z)ds = \int_a^b f(g(t),h(t),k(t))\lvert\vec{r}\,'(t)\rvert dt$	**Kurve-lengde** $\int_C ds = \int_a^b \lvert\vec{r}\,'(t)\rvert dt$
Arbeid	$W = \int_C \vec{F}\cdot\vec{T}ds = \int_C \vec{F}\cdot d\vec{r} = \int_C F_1dx + F_2dy + F_3dz$	
Strømning **Sirkulasjon**	$\int_C \vec{F}\cdot\vec{T}ds = \int_C \vec{F}\cdot d\vec{r} = \int_C F_1dx + F_2dy$ $\oint_C \vec{F}\cdot\vec{T}ds = \oint_C \vec{F}\cdot d\vec{r} = \oint_C F_1dx + F_2dy$	**I planet**
Fluks	$\int_C \vec{F}\cdot\vec{n}ds = \int_C F_1dy - F_2dx \qquad \vec{n} = \vec{T}\times\vec{k}$	**I planet**
Fundamental-teoremet for kurve-integraler	$\vec{F}$ vektorfelt med kontinuerlige komponenter over et åpent område D $\Downarrow$ $\exists$ differensierbar funksjon f : $\vec{F} = \nabla f$ $\int_A^B \vec{F}\cdot d\vec{r} = f(B) - f(A)$	
Eksakt differensialform	$F_1dx + F_2dy + F_3dz$ er en differensialform En differensialform er eksakt hvis $M\,dx + N\,dy + P\,dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$ Differensialform $F_1dx + F_2dy + F_3dz$ er eksakt $\Updownarrow$ $\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z} \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} \qquad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$	
F konservativ (vei-uavhengig)	$\int_C \vec{F}\cdot d\vec{r} = f(b) - f(a)$ $\oint_C \vec{F}\cdot d\vec{r} = 0$ $\vec{F} = \nabla f$ $\mathrm{curl}\vec{F} = \nabla\times\vec{F} = \vec{0}$ $\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z} \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} \qquad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$	$\vec{F}$ Konservativ $F_1dx + F_2dy + F_3dz$ Eksakt

Flate-integral

Flate-integral	$\iint_S g\,dS = \iint_R g(x,y,z)\frac{\lvert\nabla f\rvert}{\lvert\nabla f\cdot\vec{p}\rvert}dA$	**Flate-areal** $\iint_S dS = \iint_R \frac{\lvert\nabla f\rvert}{\lvert\nabla f\cdot\vec{p}\rvert}dA$
Parameterisert flate-integral	$\iint_S G(x,y,z)dS = \iint_R G(f(u,v),g(u,v),h(u,v))\lvert\vec{r}_u\times\vec{r}_v\rvert dudv$	**Flate-areal** $\iint_S dS = \iint_R \lvert\vec{r}_u\times\vec{r}_v\rvert dudv$
Fluks	$\iint_S \vec{F}\cdot\vec{n}dS$	
Enhetsnormalvektor	$\vec{n} = \frac{\nabla f}{\lvert\nabla f\rvert}$ ***f* skalarfunksjon som har flaten *S* som en nivåflate**	$\vec{n} = \frac{\vec{r}_u\times\vec{r}_v}{\lvert\vec{r}_u\times\vec{r}_v\rvert}$ **Parameterisert flate**

Green's teorem - Stoke's teorem

2D		3D	
Green - Normalform $\text{Divergens} = \text{Flukstetthet} = \frac{\text{Fluks}}{\text{Areal}}$	$\oint_C \vec{F}\cdot\vec{n}ds = \oint_C F_1dy - F_2dx$ $= \iint_R \left[\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}\right]dxdy$ $= \iint_R \text{div}\vec{F}dA$ $= \iint_R \nabla\cdot\vec{F}dA$	$\oiint_S \vec{F}\cdot\vec{n}dS = \oiint_S F_1dydz + F_2dzdx + F_3dxdy$ $= \iiint_D \left[\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}+\right]dxdydz$ $= \iiint_D \text{div}\vec{F}dV$ $= \iiint_D \nabla\cdot\vec{F}dV$	**Gauss Divergens**
Green - Tangensialform $Curl = \text{Sirkulasjonstetthet} = \frac{\text{Sirkulasjon}}{\text{Areal}}$	$\oint_C \vec{F}\cdot\vec{T}ds = \oint_C \vec{F}\cdot d\vec{r}$ $= \oint_C F_1dx + F_2dy$ $= \iint_R \left[\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right]dxdy$ $= \iint_R \text{curl}\vec{F}\cdot\vec{k}dA$ $= \iint_R (\nabla\times\vec{F})\cdot\vec{k}dA$	$\oint_C \vec{F}\cdot\vec{T}ds = \oint_C \vec{F}\cdot d\vec{r}$ $= \oint_C F_1dx + F_2dy + F_3dz$ $= \iint_S \left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)dydz+\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)dzdx+\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)dxdy$ $= \iint_S \text{curl}\vec{F}\cdot\vec{n}dS$ $= \iint_S \nabla\times\vec{F}\cdot\vec{n}d\sigma$	**Stoke**

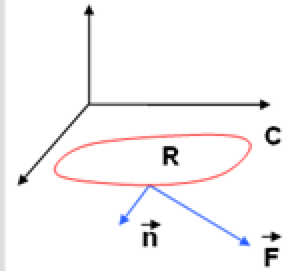
2D

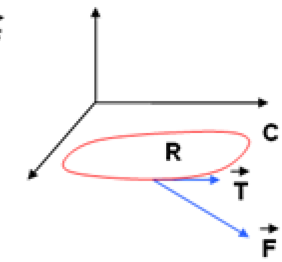
Green - Divergens

$$\Phi = \oint_C \vec{F} \cdot \vec{n} \, ds = \iint_R \nabla \cdot \vec{F} \, dA$$

Green - Curl

$$C = \oint_C \vec{F} \cdot \vec{T} \, ds = \iint_R (\nabla \times \vec{F}) \cdot \vec{k} \, dA$$





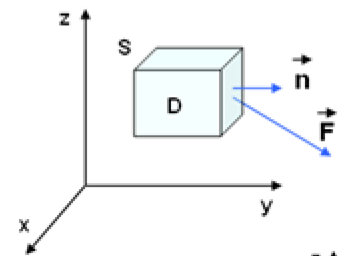
3D

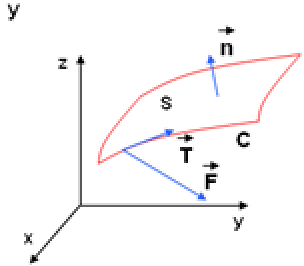
Gauss - Divergens

$$\Phi = \oiint_S \vec{F} \cdot \vec{n} \, dS = \iiint_D \nabla \cdot \vec{F} \, dV$$

Stokes - Curl

$$C = \oint_C \vec{F} \cdot \vec{T} \, ds = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS$$





Substitusjon i multiple integraler

2D	$\iint_D F(x,y)\,dxdy = \iint_D F(x(u,v), y(u,v))\,\lvert J(u,v)\rvert\,dudv$ $J(u,v) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$ $\quad J^{-1}(x,y) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$ $\quad J(u,v)J^{-1}(x,y) = 1$
3D	$\iiint_D F(x,y,z)\,dxdydz = \iiint_G F(x(u,v,w), y(u,v,w), z(u,v,w))\,\lvert J(u,v,w)\rvert\,dudvdw$ $J(u,v,w) = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$ $\quad J^{-1}(x,y,z) = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$ $\quad J(u,v,w)J^{-1}(x,y,z) = 1$

Greens form av areal

Tangensiell form	$A = \iint dA = \oint x\,dy$ $\quad \oint F_1 dx + F_2 dy = \iint \left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right] dA$ $\quad F_2 = x \quad F_1 = 0$ $A = \iint dA = -\oint y\,dx$ $\quad \oint F_1 dx + F_2 dy = \iint \left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right] dA$ $\quad F_2 = 0 \quad F_1 = -y$
Normalform	$A = \iint dA = \frac{1}{2}\oint x\,dy - y\,dx$ $\quad \oint F_1 dy - F_2 dx = \iint \left[\frac{\partial F_1}{\partial x} - \frac{\partial F_2}{\partial y}\right] dA$ $\quad F_1 = \frac{1}{2}x \quad F_2 = \frac{1}{2}y$

Masse - Masse-senter - Treghetsmoment

	Kurve	Flate	Volum
Masse	$M = \int_C \delta ds$	$M = \iint_S \delta dS$	$M = \iiint_V \delta dV$
Masse-senter	$\bar{x} = \frac{1}{M}\int_C x\delta ds$ $\bar{y} = \frac{1}{M}\int_C y\delta ds$ $\bar{z} = \frac{1}{M}\int_C z\delta ds$	$\bar{x} = \frac{1}{M}\iint_S x\delta dS$ $\bar{y} = \frac{1}{M}\iint_S y\delta dS$	$\bar{x} = \frac{1}{M}\iiint_V x\delta dV$ $\bar{y} = \frac{1}{M}\iiint_V y\delta dV$ $\bar{z} = \frac{1}{M}\iiint_V z\delta dV$
Treghetsmoment	$I_x = \int_C (y^2 + z^2)\delta ds$ $I_y = \int_C (x^2 + z^2)\delta ds$ $I_z = \int_C (x^2 + y^2)\delta ds$ $I_L = \int_C r^2 \delta ds$ $R_L = \sqrt{\frac{I_L}{M}}$	$\mathrm{I_x} = \iint_S \mathrm{y}^2 \delta \mathrm{dS}$ $\mathrm{I_y} = \iint_S \mathrm{x}^2 \delta \mathrm{dS}$ $\mathrm{I_L} = \iint_S \mathrm{r}^2 \delta \mathrm{dS}$ $\mathrm{I_o} = \iint_S (\mathrm{x}^2 + \mathrm{y}^2)\delta \mathrm{dS}$ $\mathrm{R_L} = \sqrt{\frac{\mathrm{I_L}}{\mathrm{M}}}$	$I_x = \iiint_V (y^2 + z^2)\delta dV$ $I_y = \iiint_V (x^2 + z^2)\delta dV$ $I_z = \iiint_V (x^2 + y^2)\delta dV$ $I_L = \iiint_V r^2 \delta dV$ $R_L = \sqrt{\frac{I_L}{M}}$

Fourier

Fourier-rekke (Periode 2L)	$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi t}{L}+b_n\sin\frac{n\pi t}{L}\right]$ $a_n=\frac{1}{L}\int_{-L}^{L}f(t)\cos\frac{n\pi t}{L}dt \qquad n=0,1,2,3,\ldots$ $b_n=\frac{1}{L}\int_{-L}^{L}f(t)\sin\frac{n\pi t}{L}dt \qquad n=1,2,3,\ldots$	**Alternativt** $a_n=\frac{1}{L}\int_{c}^{c+2L}f(t)\cos\frac{n\pi t}{L}dt \qquad n=0,1,2,3,\ldots$ $b_n=\frac{1}{L}\int_{c}^{c+2L}f(t)\sin\frac{n\pi t}{L}dt \qquad n=1,2,3,\ldots$
Even funksjon f(-t) = f(t)	$a_n=\frac{2}{L}\int_{0}^{L}f(t)\cos\frac{n\pi t}{L}dt \qquad n=0,1,2,3,\ldots$ $b_n=0 \qquad n=1,2,3,\ldots$	
Odd funksjon f(-t) = -f(t)	$a_n=0 \qquad n=0,1,2,3,\ldots$ $b_n=\frac{2}{L}\int_{0}^{L}f(t)\sin\frac{n\pi t}{L}dt \qquad n=1,2,3,\ldots$	

Differensial-ligninger

Svinge-ligning	$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=F$	$mx_{tt}+cx_t+kx=F$
Bølge-ligning	$\frac{\partial^2 y}{\partial t^2}=v^2\frac{\partial^2 y}{\partial x^2}$	$y_{tt}=v^2 y_{xx}$
Varme-ligning	$\frac{\partial u}{\partial t}=\kappa\nabla^2 u$	$u_t=\kappa y_{xx}$

Svinge-ligning m / initialbetingelser

Kloss + fjær uten dempning	$mx''+kx=F(t)$	$0 \le t \le L$ $x(0)=0$ $x(L)=0$
	$x(t)=x_0(t)+x_p(t)=c_1\cos(\omega_0 t)+c_2\sin(\omega_0 t)+x_p$ $F(t)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L}$ $b_n=\frac{2}{L}\int_0^L F(t)\sin\frac{n\pi t}{L}\,dt$ $x(t)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi t}{L}$ $=\sum_{n=1}^{\infty}\frac{1}{k-m\frac{n^2\pi^2}{L^2}} b_n \sin\frac{n\pi t}{L}$	$\omega_0=\sqrt{\frac{k}{m}}$
Kloss + fjær med dempning	$mx''+cx'+kx=F(t)$	$0 \le t \le L$ $x(0)=0$ $x(L)=0$
	$x_p(t)=\frac{F_0}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}}\sin(\omega t-\alpha)$ $\alpha=\tan^{-1}\frac{c\omega}{k-m\omega^2}$	$F(t)=F_0\sin(\omega t)$
	$F(t)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L}$ $b_n=\frac{2}{L}\int_0^L F(t)\sin\frac{n\pi t}{L}\,dt$ $x(t)=\sum_{n=1}^{\infty}\frac{b_n\sin(\omega_n t-\alpha_n)}{(k-m\omega_n{}^2)^2+(c\omega_n)^2}$	

Varme- og bølge-ligning m / initialbetingelser

Varme-ligning **m / initialbetingelser** **Fourier sinus-rekke**	$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad 0 < x < L$	$u(0,t)=0$ $u(L,t)=0$ $u(x,0)=f(x)$
	$u(x,t)=\sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2 k}{L^2}t}\sin\frac{n\pi x}{L}$ $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$	
Bølge-ligning **m / initialbetingelser** **Fourier sinus-rekke**	$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} \qquad 0 < x < L$	$y(0,t)=0$ $y(L,t)=0$ $y(x,0)=f(x)$ $y_t(x,0)=g(x)$
	$y(x,t)=\sum_{n=1}^{\infty} A_n \cos\frac{n\pi at}{L}\sin\frac{n\pi x}{L}$ $A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$	**Problem A** $y(0,t)=0$ $y(L,t)=0$ $y(x,0)=f(x)$ $y_t(x,0)=0$
	$y(x,t)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi at}{L}\sin\frac{n\pi x}{L}$ $B_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$	**Problem B** $y(0,t)=0$ $y(L,t)=0$ $y(x,0)=0$ $y_t(x,0)=g(x)$
	$y(x,t)=y_A(x,t)+y_B(x,t)$	